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On the possibility of breakdown of Lorentz invariance in the Takahashi–Umezawa quantization method

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Abstract. The Stückelberg formalism, for a massive spin-one field interacting with external potentials, is quantized following the method of Takahashi and Umezawa. It is shown how the characteristic determinant of the field equations, which, in an earlier paper, was shown to have the same form as the Lee–Yang determinant, for such a theory, determines the possibility of breakdown of Lorentz invariance in the procedure of the Takahashi–Umezawa quantization method.

1. Introduction

In earlier papers (Jenkins 1973a, b, c) the relationship between the causal nature of a classical relativistic field theory (Velo and Zwanziger 1969a, b, 1971) and the Lorentz invariance of the corresponding quantum theory, was explored in detail for a massive spin-one field.

For convenience of calculation, the most important result of the above papers (Jenkins 1973c) was derived in the formalism due to Stückelberg (1938). Spin one was described in terms of the Stückelberg fields $A_\mu(x)$ and $\Theta(x)$, and attention was restricted to interaction lagrangians of the form

$$\mathscr{L}_I(x) = \mathscr{L}_I\left(A_\mu(x)+\frac{1}{m}\partial_\mu\Theta(x), \partial_\lambda A_\rho(x)-\partial_\rho A_\lambda(x), \text{external potentials}\right) \quad (1)$$

which satisfy the requirement of being quadratic in $\partial_0 A_\mu(x)$ and $\partial_0\Theta(x)$. In these circumstances, it was shown that the classical theory is causal, in the sense of Velo and Zwanziger, if and only if the corresponding quantum theory is Lorentz invariant, in that the perturbation expansion for the S operator in the interaction picture is normal-independent.

Central to the proof of this result is that the characteristic determinant, which determines the causal nature of the classical theory, and the Lee–Yang determinant (Lee and Yang 1962), which determines the normal-dependence of the perturbation expansion for the S operator in the interaction picture, have identical form.

In the present paper, the Stückelberg formalism is again used to demonstrate that the characteristic (Lee–Yang) determinant has further importance, in that it plays a significant role in the quantization method of Takahashi and Umezawa (1953) (Takahashi 1969).

2. Takahashi–Umezawa quantization

In order to facilitate the calculations of the next section, a brief resumé of the relevant parts of the Takahashi–Umezawa quantization method is given. For a fuller discussion, the book of Takahashi (1969) should be consulted.

A free, multicomponent field, $\phi(x)$, is assumed to satisfy the homogeneous linear wave equation

$$\Lambda(\partial)\phi(x) = 0. \tag{2}$$

The corresponding Klein–Gordon divisor, $d(\partial)$, is defined by

$$\Lambda(\partial)\, d(\partial) = d(\partial)\Lambda(\partial) = (\partial^2 + m^2)1, \tag{3}$$

where m is the mass of the field and 1 is the metric matrix in the index space of $\phi(x)$.

In the presence of interaction, the field equations for the Heisenberg picture field become

$$\Lambda(\partial)\phi(x) = J(x) + \partial^\mu K_\mu(x), \tag{4}$$

where

$$J(x) = -\frac{\partial \mathscr{L}_I}{\partial \phi}(x) \qquad \text{and} \qquad K_\mu(x) = \frac{\partial \mathscr{L}_I}{\partial \partial^\mu \phi}(x), \tag{5}$$

and where the interaction lagrangian, $\mathscr{L}_I(x)$, has, sufficiently for the present purposes, been assumed to depend only on $\phi(x)$, $\partial_\mu \phi(x)$ and external potentials.

Next (4) is solved by the method of Green's function giving

$$\phi(x) = \phi_0(x) - \int_{-\infty}^{\infty} d(\partial)[\Delta^{\text{ret}}(x-x')J(x') + \partial^\mu \Delta^{\text{ret}}(x-x')K_\mu(x')]\, \mathrm{d}^4x', \tag{6}$$

where $\phi_0(x)$ is a solution of the free-field equation (2) and

$$\Delta^{\text{ret}}(x) = \theta(x_0)\Delta(x)$$

with $\theta(x_0)$ the unit step function and $\Delta(x)$ the usual solution of the Klein–Gordon equation.

Auxiliary fields are now introduced, and, for a given spacelike surface σ, are defined by

$$\phi(x, \sigma) = \phi_0(x) - \int_{-\infty}^{\sigma} d(\partial)[\Delta(x-x')J(x') + \partial^\mu \Delta(x-x')K_\mu(x')]\, \mathrm{d}^4x'. \tag{7}$$

The auxiliary fields satisfy both the free-field equation (2) and the corresponding free-field commutation relations. Equations (6) and (7) now give, for x on σ (denoted by x/σ),

$$\begin{aligned}\phi(x) = \phi(x/\sigma) + \int_{-\infty}^{\infty} \{&[\theta(x_0 - x_0'), d(\partial)]\Delta(x-x')J(x') \\ &+ [\theta(x_0 - x_0'), \partial^\mu d(\partial)]\Delta(x-x')K_\mu(x')\}\, \mathrm{d}^4x'.\end{aligned} \tag{8}$$

In similar fashion, it follows that

$$\begin{aligned}\partial_\lambda \phi(x) = \partial_\lambda \phi(x/\sigma) + \int_{-\infty}^{\infty} \{&[\theta(x_0 - x_0'), \partial_\lambda d(\partial)]\Delta(x-x')J(x') \\ &+ [\theta(x_0 - x_0'), \partial^\mu \partial_\lambda d(\partial)]\Delta(x-x')K_\mu(x')\}\, \mathrm{d}^4x'\end{aligned} \tag{9}$$

etc. These equations (8) and (9) now reduce to expressions of the form

$$\phi(x) = \phi(x/\sigma) + g(n, \partial)J(x) + h_\mu(n, \partial)K^\mu(x) \tag{10}$$

$$\partial_\lambda \phi(x) = \partial_\lambda \phi(x/\sigma) + g_\lambda(n, \partial)J(x) + h_{\lambda\mu}(n, \partial)K^\mu(x) \tag{11}$$

respectively etc, where n_μ is the unit normal to the spacelike surface σ.

It is this stage, in the Takahashi–Umezawa quantization method, which is critical for the discussions of the present paper. For, in order to proceed with the quantization and calculate the interaction hamiltonian in the interaction picture, (10) and (11) must be solved for $\phi(x)$ and $\partial_\mu \phi(x)$ in terms of the auxiliary fields and their derivatives evaluated for x on σ.

It will be seen, in the next section, for quite general interactions of the Stückelberg field in external potentials, that the possibility of doing this depends on the possibility of inverting a matrix, the determinant of which has the same form as the characteristic (Lee–Yang) determinant.

3. Stückelberg formalism

The lagrangian for the interacting Stückelberg field is assumed to have the form

$$\mathscr{L}(x) = \mathscr{L}_0(x) + \mathscr{L}_I(x) \tag{12}$$

with the free lagrangian given by

$$\mathscr{L}_0(x) = -\tfrac{1}{2}\partial_\mu A_\nu(x)\partial^\mu A^\nu(x) + \tfrac{1}{2}m^2 A_\mu(x)A^\mu(x) + \tfrac{1}{2}\partial_\mu \Theta(x)\partial^\mu \Theta(x) - \tfrac{1}{2}m^2\Theta^2(x) \tag{13}$$

and the interaction lagrangian (1) of the form

$$\mathscr{L}_I(x) = \tfrac{1}{2}A_{kj}(x)(\partial_k A_0(x) - \partial_0 A_k(x))(\partial_j A_0(x) - \partial_0 A_j(x)) + B_k(x)\left(A_0(x) + \frac{1}{m}\partial_0\Theta(x)\right)$$
$$\times(\partial_k A_0(x) - \partial_0 A_k(x)) + \tfrac{1}{2}C(x)\left(A_0(x) + \frac{1}{m}\partial_0\Theta(x)\right)^2 + D(x) \tag{14}$$

where $A_{kj}(x)$, $B_k(x)$, $C(x)$ depend only on $A_k(x) + (1/m)\partial_k\Theta(x)$, $\partial_k A_j(x) - \partial_j A_k(x)$ and the external potentials, whilst, in addition to this dependence, $D(x)$ also depends linearly on $A_0(x) + (1/m)\partial_0\Theta(x)$ and $\partial_k A_0(x) - \partial_0 A_k(x)$, and where repeated latin indices are summed over the values 1, 2, 3. The lack of manifest Lorentz invariance in the (formally invariant) expression (14) will be convenient for the following calculation.

In order to write down the expressions, for the interacting Stückelberg field theory defined by (12), (13) and (14), which correspond to (10) and (11), it is convenient to specialize to a flat spacelike surface with unit normal $n_\mu = (1, 0, 0, 0)$, when it is sufficient to note the following.

The Klein–Gordon divisors of $A_\mu(x)$ and $\Theta(x)$ are given respectively by

$$d_{\mu\nu}(\partial) = g_{\mu\nu} \qquad \text{and} \qquad d(\partial) = -1 \tag{15}$$

$$\frac{\partial \mathscr{L}_I}{\partial \partial^0 A^0}(x) = 0 \tag{16}$$

$$\frac{\partial \mathscr{L}_I}{\partial \partial^0 A^k}(x) = \tfrac{1}{2}(A_{kj}(x)+A_{jk}(x))(\partial_j A_0(x)-\partial_0 A_j(x))+B_k(x)\left(A_0(x)+\frac{1}{m}\partial_0\Theta(x)\right)+\frac{\partial D}{\partial\partial^0 A^k}(x) \tag{17}$$

$$\frac{\partial \mathscr{L}_I}{\partial \partial^0 \Theta}(x) = \frac{B_k(x)}{m}(\partial_k A_0(x)-\partial_0 A_k(x))+\frac{C(x)}{m}\left(A_0(x)+\frac{1}{m}\partial_0\Theta(x)\right)+\frac{\partial D}{\partial\partial^0 \Theta}(x) \tag{18}$$

and finally

$$[\theta(x_0-x_0'),\partial_0]\Delta(x-x') = 0 \tag{19}$$

$$[\theta(x_0-x_0'),\partial_0^2]\Delta(x-x') = \delta^4(x-x'). \tag{20}$$

Now remembering (5) and using (15)–(20) in the expressions corresponding to (8) and (9) gives

$$A_\mu(x) = A_\mu(x/\sigma), \qquad \partial_k A_\mu(x) = \partial_k A_\mu(x/\sigma), \qquad \partial_0 A_0(x) = \partial_0 A_0(x/\sigma),$$
$$\partial_k\Theta(x) = \partial_k\Theta(x/\sigma); \tag{21}$$

$$\partial_0 A_k(x) = \partial_0 A_k(x/\sigma)+\tfrac{1}{2}(A_{kj}(x)+A_{jk}(x))(\partial_j A_0(x)-\partial_0 A_j(x))$$
$$+B_k(x)\left(A_0(x)+\frac{1}{m}\partial_0\Theta(x)\right)+\frac{\partial D}{\partial\partial^0 A^k}(x); \tag{22}$$

$$\partial_0\Theta(x) = \partial_0\Theta(x/\sigma)-\frac{B_k(x)}{m}(\partial_k A_0(x)-\partial_0 A_k(x))-\frac{C(x)}{m}\left(A_0(x)+\frac{1}{m}\partial_0\Theta(x)\right)-\frac{\partial D}{\partial\partial^0\Theta}(x). \tag{23}$$

Since (21) are already in the desired form, it only remains to use these in (22) and (23) to facilitate the writing of $\partial_0 A_k(x)$ and $\partial_0\Theta(x)$ in terms of the auxiliary fields and their derivatives. Thus, on remembering the assumed dependence of $A_{ij}(x)$, $B_k(x)$, $C(x)$, $D(x)$ on the Heisenberg picture field variables, (22) and (23) may be written respectively as

$$\partial_0 A_k(x) = \partial_0 A_k(x/\sigma)+\tfrac{1}{2}(A_{kj}(x/\sigma)+A_{jk}(x/\sigma))(\partial_j A_0(x/\sigma)-\partial_0 A_j(x))$$
$$+B_k(x/\sigma)\left(A_0(x/\sigma)+\frac{1}{m}\partial_0\Theta(x)\right)+\frac{\partial D}{\partial\partial^0 A^k}(x/\sigma); \tag{24}$$

$$\partial_0\Theta(x) = \partial_0\Theta(x/\sigma)-\frac{B_k(x/\sigma)}{m}(\partial_k A_0(x/\sigma)-\partial_0 A_k(x))-\frac{C(x/\sigma)}{m}\left(A_0(x/\sigma)+\frac{1}{m}\partial_0\Theta(x)\right)$$
$$-\frac{\partial D}{\partial\partial^0\Theta}(x/\sigma). \tag{25}$$

Now (24) and (25) provide a set of simultaneous linear equations for the Heisenberg fields, with coefficients dependent only on the auxiliary fields, their derivatives and the external potentials. To solve these equations for $\partial_0 A_k(x)$ and $\partial_0\Theta(x)$, it is necessary to invert a matrix, the determinant of which is given by

$$\mathscr{D} \equiv \begin{vmatrix} \delta_{ij}+\tfrac{1}{2}(A_{ij}(x/\sigma)+A_{ji}(x/\sigma)) & -\dfrac{B_i(x/\sigma)}{m} \\ -\dfrac{B_j(x/\sigma)}{m} & 1+\dfrac{C(x/\sigma)}{m^2} \end{vmatrix} \tag{26}$$

Note that $\mathscr{D}$ has exactly the same form as the expression given by Jenkins (1973c) for the characteristic (Lee–Yang) determinant. For the ensuing discussions, it is more convenient that (26) is generalized to an arbitrary spacelike surface σ, with unit normal n_μ. (26), thus generalized, will be denoted by $\mathscr{D}(n)$, and is just of the same form as the corresponding generalization of the characteristic (Lee–Yang) determinant.

4. Discussion

Firstly it is noted that a relativistic quantum field theory may be constructed from the lagrangian given by (12), (13) and (14) only if the consequent field equations form a hyperbolic system. Thus, in the following, only couplings for which the field equations are hyperbolic will be considered. This means that the characteristic determinant $\mathscr{D}(n)$ is restricted to having only real roots n_μ (Courant and Hilbert 1962).

In the preceding pages, it has been seen that, for quantization on a given spacelike surface σ, the Takahashi–Umezawa method proceeds unhindered if and only if the characteristic (Lee–Yang) determinant $\mathscr{D}(n)$ does not vanish for n_μ normal to σ. Now since the normal to a three-dimensional surface is timelike if and only if the surface is spacelike, it follows that the Takahashi–Umezawa quantization method may be formally carried through for *all* spacelike surfaces if and only if $\mathscr{D}(n)$ has no real timelike root. However, since the possibility of complex roots was excluded from the outset, this condition on the roots of $\mathscr{D}(n)$ is necessary and sufficient for Lorentz invariance of the present type of theory (Jenkins 1973c).

This result provides a simple view of how a breakdown of Lorentz invariance can occur in the Takahashi–Umezawa quantization method, for the present type of theory. For if $\mathscr{D}(n)$ were to have a real timelike root, there would be a corresponding class of spacelike surfaces, to which the root would be normal. Then, by the above, the Takahashi–Umezawa quantization method could not be carried through on these spacelike surfaces. Thus a breakdown of Lorentz invariance would manifest itself as a dependence of the *procedure* of the Takahashi–Umezawa method on the quantization surface.

Usually the possibility of breakdown of Lorentz invariance in quantum field theories, derived from apparently invariant lagrangians, is explored by inspection of Dyson's formula for the S operator, in the interaction picture, modified according to the Lee–Yang theorem (Matthews 1949, Lee and Yang 1962, Suzuki and Hattori 1972, Kvitky and Mouton 1972). By contrast, the present work isolates that stage in the procedure of the Takahashi–Umezawa quantization method at which a possible breakdown of Lorentz invariance would first manifest itself; namely the point at which it is attempted to express the Heisenberg picture fields in terms of the auxiliary fields.

Although the present note is only concerned with a massive spin-one Stückelberg field interacting (quite generally) with external potentials, the result that the characteristic (Lee–Yang) determinant plays a significant role in the Takahashi–Umezawa quantization method, and the nature of that role, are expected to be the same for fields of arbitrary spin.

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